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LETTER TO THE EDITOR

Soliton point particles of extended evolution equations

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Abstract. A Hamiltonian formulation of the particle representation of solitons of extended hierarchies of evolution equations is presented. The extension is related to the addition of new terms with x-dependent coefficients.

In a previous paper [1], we presented a method for finding the point representation of soliton solutions of the hierarchies of non-linear evolution equations generated by an appropriate recursion operator. The method was based on the concept of master symmetries [2] and each solution of the considered hierarchy was of the form $u_t = K_n(a, u)$, with $K_n = K_n(a, u, u_x, \ldots)$, where $u = u(x, t)$ and a was a constant.

In this letter we consider the following generalised hierarchies of evolution equations:

$$u_t = K_n(a, u) + b\tau(x, u) \tag{1}$$

where $\tau(x, u)$ is the simplest master symmetry of degree 1 that is not a constant and b is an arbitrary constant. We confine our considerations to Korteweg-de Vries (KdV), Sawada-Kotera-Caudray-Dodd-Gibbon (SKCDG), modified KdV (MKdV) and Boussinesq (B) hierarchies, but the method may be applied to other soliton hierarchies as well.

The extended KdV hierarchy (EKdV) has the form

$$\begin{aligned} &u_t = \phi^n u_x + b(xu_x + 2u) = \theta\delta I^{(n)}/\delta u \\ &I^{(n)} = H^{(n-1)} + b\delta_1 \qquad n = 0, 1, \ldots \end{aligned} \tag{2}$$

where ϕ is the recursion operator, $\tau = xu_x + 2u$ is the first non-constant master symmetry, $H^{(n-1)}$ is the functional connected with the symmetry $\phi^n u_x$, δ_1 is a master functional connected with the master symmetry τ and $\theta = D^3 + \frac{2}{3}auD + \frac{1}{3}au_x$ is the second implectic operator of the KdV hierarchy, as the simplest one, $\theta = D$, leads to a constant master symmetry.

According to (4.9), (4.14) and (4.16) from [1], for pure soliton solutions with $a = 6$, we find the soliton-particle Hamiltonian for the $(n+1)$th evolution equation (2) in the form

$$I^{(n+1)} = \sum_{i=1}^{N} \left(\frac{1}{2n+1} \tilde{p}_i^{2n+1} - b\tilde{p}_1\tilde{q}_i \right) \qquad \tilde{p}_i = 2\kappa_i. \tag{3}$$

Note that our particle variables are connected with the second Hamiltonian structure of the hierarchy, contrary to those from [1] which are connected with the first Hamiltonian structure. The equations of motion for the particle variables $(\tilde{p}_i, \tilde{q}_i)$ of the

N-soliton solution of the $(n+1)$th equation from the hierarchy (2) are

$$\dot{\tilde{p}}_i = \{\tilde{p}_i, I^{(n+1)}\}_{\tilde{\theta}} = -\partial I^{(n+1)}/\partial \tilde{q}_i = b\tilde{p}_i$$
$$\dot{\tilde{q}}_i = \{\tilde{q}_i, I^{(n+1)}\}_{\tilde{\theta}} = \partial I^{(n+1)}/\partial \tilde{p}_i = \tilde{p}_i^{2n} - b\tilde{q}_i \tag{4}$$

where $\bar{\theta}$ is the standard implectic operator from classical mechanics. The appropriate equations in the space of functionals are

$$\dot{P} = \{P, I^{(n+1)}\}_\theta = bP \qquad \dot{\delta}_1 = \{\delta_1, I^{(n+1)}\}_\theta = (2n+1)H^{(n)} \tag{5}$$

and are connected with (4) through the map $\delta : F \to \tilde{F}$ of functionals into their values.

Applying the results of [1] we have obtained the following point-particle Hamiltonians of the N-soliton solution of various extended hierarchies (1) of evolution equations.

(i) Two hierarchies of the extended SKCDG equation in the frame of the first Hamiltonian structure and $a^2 = 12$:

$$I^{(n,1)} = \sum_i \left(\frac{1}{6n+1} p_i^{6n+1} - bp_iq_i\right)$$
$$I^{(n,2)} = \sum_i \left(\frac{1}{6n+5} p_i^{6n+5} - bp_iq_i\right). \tag{6}$$

(ii) Two hierarchies of the extended Boussinesq equation in the frame of the second Hamiltonian structure and $a^2 = 72$:

$$I^{(n+1,1)} = \sum_i \left((\tfrac{4}{3})^n (\mathrm{sgn}\,\theta_i)^n \frac{1}{3n+1} p_i^{3n+1} + (\mathrm{sgn}\,\mu_i) bp_iq_i\right) \tag{7a}$$

$$I^{(n+1,2)} = \sum_i \left((\tfrac{4}{3})^n (\mathrm{sgn}\,\mu_i)^{n+1} \frac{1}{3n+2} p_i^{3n+2} + (\mathrm{sgn}\,\mu_i) bp_iq_i\right) \tag{7b}$$

where $\mathrm{sgn}\,\mu_i$ fixes the direction of soliton motion.

(iii) The hierarchy of the extended MKdV equation in the frame of the first Hamiltonian structure and $a = 6$:

$$I^{(n)} = \sum_i \left(\frac{1}{2n+1} p_i^{2n+1} - bp_iq_i\right) \qquad n = \ldots -1, 0, 1, \ldots. \tag{8}$$

In all above examples $p_i = 2\kappa_i$ and to simplify the notation we have dropped the tilde.

Equations of motion for ith soliton particle, according to (4), take the form

$$\dot{p}_i = \alpha p_i \qquad \dot{q}_i = \beta p_i^k - \alpha q_i \tag{9}$$

with various values of α, β and k. For example, the EMKdV hierarchy has $\alpha = b$, $\beta = 1$, $k = 2n$ and the first EB hierarchy has $\alpha = -(\mathrm{sgn}\,\mu_i)b$, $\beta = (\frac{4}{3})^n(\mathrm{sgn}\,\mu_i)^n$, $k = 3n$. The solutions of (9) are

$$p_i(t) = p_i(0)\exp(\alpha, t) \qquad q_i(t) = \frac{\beta p_i^k(0)}{\alpha(k+1)} \exp(-\alpha t)\{\exp[\alpha(k+1)t] - 1\}. \tag{10}$$

The one-soliton solution for EKdV, ESKCDG and EB hierarchies has the form

$$u_s = (3/a)p^2(t)\,\mathrm{sech}^2[\tfrac{1}{2}p(t)(x + q(t))] \tag{11}$$

with the appropriate value of the coefficient a. The form of the solution (11) may be verified by inspection. The velocity of this soliton is

$$v(t)=\dot{q}(t)=\frac{\beta p^{k}(0)}{k+1}\exp(-\alpha t)\{1+k\exp[\alpha(k+1)t]\} \tag{12}$$

hence, for $t\to\pm\infty$ $v(t)\to\infty$ and for $t=0$ $v(t)$ reaches the value characteristic for a common soliton. Moreover, for $\alpha<0$ and $t\to-\infty$ the amplitude of the soliton diverges and its width vanishes, and for $t\to+\infty$ the amplitude of the soliton vanishes and the width diverges. For $\alpha>0$ the situation is reversed. The area of the soliton evolves in time proportionally to $p(t)$: $\int_{-\infty}^{\infty} u_s\,\mathrm{d}x \sim p(t)$.

The one-soliton solution for the EMKdV hierarchy has the form

$$u_s=p(t)\,\mathrm{sech}\{p(t)[x+q(t)]\} \tag{13}$$

with time evolution similar to the evolution of (11), but now the area of (13) is preserved in time.

EKdV and EMKdV hierarchies are considered by Calogero and Degasperies [3, 4] on the basis of the inverse scattering method, but the other results, as well as the Hamiltonian formulation of soliton-particle dynamics, are new.

References

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